THE SOLUTION OF A TYPE OF TWO-DIMENSIONAL INTEGRAL EQUATIONS

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Many spatial mixed (boundary value) problems in the theory of elasticity (for example, the contact problem for an elastic layer) may be reduced to the solution of an integral equation of the form

$$\int_\Omega \varphi(P)\, K\left(\frac{R}{h}\right) dP = 2\pi h f(Q) \qquad \left(\begin{matrix} Q \in \Omega \\ h \in (0, \infty), \end{matrix} \quad R = \sqrt{(x-\xi)^2 + (y-\eta)^2}\right) \tag{1}$$

where Q is a point with coordinates (x, y) and P is a point with coordinates (ξ, η). The kernel of Equation (1) is given by the Formula

$$K(k) = \int_0^\infty L(u)\, J_0(uk)\, du \qquad \left(k = \frac{R}{h}\right) \quad (J_0(x) \text{ is the Bessel function}) \tag{2}$$

and, moreover, $L(u)$ is a continuous function and

$$L(u) \to 1 \quad \text{as} \quad u \to \infty, \qquad L(u) \to Au \quad \text{as} \quad u \to 0 \qquad (A = \text{const}) \tag{3}$$

As regards the region Ω we shall assume that it is simply-connected, convex, and bounded by a sufficiently-smooth curve L.

Let us introduce into consideration the parameter $\lambda = h/\rho$ (where $\rho = {}^1/_2 \max_\Omega R$). When the value of this parameter $\lambda > 1$, the solution of Equation (1) can be obtained sufficiently simply if we know the solution of the auxiliary integral equation [1]

$$\int_\Omega \frac{\varphi(P)}{R} dP = 2\pi f(Q), \qquad Q \in \Omega \tag{4}$$

However, the method which is so useful in that circumstance, is either not very effective or is completely inapplicable for values of $\lambda < 1$.

Of help here is only the fact that for very small values of λ we can find [1] a very simple degenerate solution of Equation (1),

$$\varphi(Q) = \frac{f(Q)}{Ah}, \qquad Q \in \Omega \tag{5}$$

In practice this solution can be used for a sufficiently wide range of variation of parameter λ, namely for $\lambda \leqslant \lambda_0$ (see below for a determination of the magnitude of λ_0).

It should now be noted that the degenerate solution (5) has one essential deficiency. Namely, it can be shown that for any value of the parameter $\lambda \in (0, \infty)$ the solution of Equation (1) should have a singularity of the type $1/\sqrt{R}$ on the contour L of region Ω From this it follows that the degenerate solution (5) will give incorrect results for values of close to L Hence, we set ourselves the task of obtaining a practical, convenient, approximate solution of Equation (1) which will be suitable for the values $0 < \lambda \leqslant 1$ and which will have a singularity of the form $1/\sqrt{R}$ on contour L .

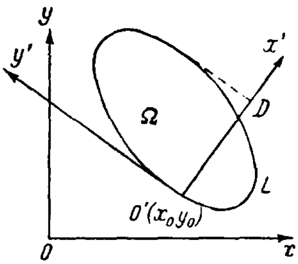

Fig. 1

Keeping in mind the first of formulas (3.10) in [2], we shall strive to obtain the indicated solution of Equation (1) for the case $f(Q) \equiv \mu$ (μ = const). Obviously, this solution should have the form

$$\varphi(Q) = \frac{\mu}{Ah}\,[1 + \Phi(Q)] \tag{6}$$

where the function $\Phi(Q)$ has the singularity of the type $1/\sqrt{R}$ on L and rapidly approaches zero as the point Q moves away from the contour L

Thus, the matter reduces to the finding of a function $\Phi(Q)$ with the stated properties.

Let $y = \omega(x)$ be the equation of contour L of region Ω , and let O' be the point on L with the coordinates x_0, y_0 Let us transfer to a new Cartesian coordinate system with origin at the point O' and, moreover, let the axis $O'y'$ be directed along the tangent to contour L and axis $O\,x$ toward the interior of region Ω

The transformation from the old coordinates x, y to the new x can be effected by Formula

$$x' = \pm \frac{1}{\sqrt{1 + \omega'^2(x_0)}}\,[(x - x_0)\,\omega'(x_0) + \omega(x_0) - y] \tag{7}$$

The formula for transforming from x, y to y' is not needed for what follows. Let d be the length of the straight line $O'D$ and let the equation of the contour in the new coordinate system have the form

$$y = \begin{cases} \psi_1(x') & (y' < 0) \\ \psi_2(x') & (y' > 0) \end{cases} \tag{8}$$

then, in the dimensionless variables

$$a = \frac{x}{h}, \qquad b = \frac{y}{h}, \qquad \alpha = \frac{\xi}{h}, \qquad \beta = \frac{\eta}{h} \tag{9}$$

Equation (1) can be represented in the form

$$\int\limits_0^{d/h} d\alpha \int\limits_{\psi_1/h}^{\psi_2/h} \varphi_*(\alpha, \beta)\, K\left(\sqrt{(a-\alpha)^2 + (b-\beta)^2}\right) d\beta = \frac{2\pi\mu}{h} \qquad ((a, b) \in \Omega_*) \tag{10}$$

Following paper [3] (*) we rewrite Equation (10) in the form

$$\int\limits_0^{\infty} d\alpha \int\limits_{-\infty}^{\infty} \varphi_*(\alpha, \beta)\, K\left(\sqrt{(a-\alpha)^2 + (b-\beta)^2}\right) d\beta = \frac{2\pi\mu}{h} + F(\varphi_*, h) \tag{11}$$

*) Note that in formulas (14) and (15) of paper [3], instead of $\Phi(D)$ there should be $\Phi(D/2)$. In connection with this, certain values in Table 1 of this paper will be changed.

where $F(\varphi_*, h)$ denotes a certain integral operator on φ_*, whose expression is not needed for what follows; let us note only that $F(\varphi_*, 0) \equiv 0$

Solving Eluation (11) for small values of λ by the method of sequential approximations, with sufficient accuracy we can restrict ourselves to the zero-th approximation which is determined from Equation

$$\int_0^\infty d\alpha \int_{-\infty}^\infty \varphi_*(\alpha, \beta)\, K \sqrt{(a-\alpha)^2 + (b-\beta)^2})\, d\beta = \frac{2\pi\mu}{h} \tag{12}$$

It is easy to show that the solution of Equation (12) will be a function of α alone, i.e. $\varphi_*(\alpha, \beta) \equiv \varphi_*(\alpha)$; without loss of generality, Equation (12) can be rewritten in the form

$$\int_0^\infty \varphi_*(\alpha)\, K_*(a-\alpha)\, d\alpha = \frac{\pi\mu}{h}, \qquad a > 0 \tag{13}$$

Here

$$K_*(a-\alpha) = \frac{1}{2}\int_{-\infty}^\infty K(\sqrt{(a-\alpha)^2+\beta^2})\, d\beta =$$

$$= \int_0^\infty L(u)\, du \int_0^\infty J_0(u\sqrt{(a-\alpha)^2+\beta^2})\, d\beta = \int_0^\infty \frac{L(u)}{u}\cos(a-\alpha)u\, du \tag{14}$$

The solution of Equation (13) can be obtained in closed form by the Wiener-Hopf method; however, we can indicate an approximate solution which is computationally convenient.

Using the properties (3) of the function $L(u)$, with a sufficient degree of accuracy (as examples show), we can approximate kernel (14) by the expression [4]

$$K_*(a-\alpha) = \int_0^\infty \frac{\cos(a-\alpha)u}{\sqrt{u^2+A^{-2}}}\, du \tag{15}$$

Then, the solution of Equation (13), corresponding to an approximate solution of Eluation (1) for small λ, can be found sufficiently easily by the Wiener-Hopf method and has the form [3] (here $\Phi(x)$ is the probability integral)

$$\varphi_*(a) = \varphi(x, y) = \frac{\mu}{Ah}\left[\Phi\left(\sqrt{\frac{2a}{A}}\right) + \sqrt{\frac{A}{\pi a}}\exp\frac{-a}{A}\right] \tag{16}$$

$$a = \frac{x'}{h} = \pm\frac{1}{h\sqrt{1+\omega'^2(x_0)}}\left[(x-x_0)\,\omega'(x_0) + \omega(x_0) - y\right] \tag{17}$$

Thus, the approximate solution in form (6) of Equation (1) is given by Formulas (16) and (17).

If the region Ω is a circle of radius ρ, then Formula (17) becomes

$$a = (\rho - r)/h \tag{18}$$

Substituting (18) into (16) we get the approximate solution for this case in the form

$$\varphi(r) = \frac{\mu}{Ah}\left[\Phi\left(\sqrt{\frac{2(\rho-r)}{Ah}}\right) + \sqrt{\frac{Ah}{\pi(\rho-r)}}\exp\frac{-(\rho-r)}{Ah}\right] \tag{19}$$

Let us determine the quantity

$$P = 2\pi\int_0^\rho \varphi(r)\, r\, dr = 4\mu\rho\varkappa =$$

$$= \frac{\pi\mu\rho}{A\lambda}\left\{\Phi\left(\sqrt{\frac{2}{A\lambda}}\right)\left[\left(1+\frac{A\lambda}{2}\right)^2 - \frac{A^2\lambda^2}{2}\right] + \sqrt{\frac{A\lambda}{\pi}}\left(1+\frac{A\lambda}{2}\right)\exp\frac{-1}{A\lambda}\right\} \tag{20}$$

For the limits of validity of the degenerate solution (5) we have

$$\frac{P - P_b}{P_b} 100\% \leqslant 5\% \left(P_b = \frac{\pi\mu\rho}{A\lambda}\right), \qquad \lambda_0 = 0.05A^{-1} \tag{21}$$

As an actual example let us consider the problem of the action of a circular plane die on an elastic layer situated without friction on a rigid foudation. The frictional force between the die and the layer is assumed to be absent.

This problem can be reduced to the solution of an integral equation of type (1) in which $\varphi(P)$ is the contact pressure between the die and the layer, h is the thickness of the layer [1],

$$f(Q) = \Delta\delta, \qquad \Delta = \frac{E}{2(1 - \sigma^2)}, \qquad L(u) = \frac{\cosh 2u - 1}{\sinh 2u + 2u} \tag{22}$$

and, δ is the sag in the layer boundary. Noting that $A = \frac{1}{2}$, from Formula (21) we have $\lambda_0 = 0.1$, i.e. the degenerate solution (5) can be used when $\lambda \leqslant 0.1$.

A more accurate solution, determined from Formulas (19) and (20), gives practically correct results when $\lambda \leqslant 1$ as was shown by the calculations which were carried out. Let us cite some results of the computation of $\varkappa$, and, for comparison, the values of the quantity as obtained in [5] by another method (the differences do not exceed 4.5%).

$1/\lambda =$	1	1.5	2	2.5	3
$\varkappa =$	2.26	3.08	3.88	4.67	5.46
$\varkappa_{[5]} =$	2.20	2.95	3.72	4.49	5.26

Let us note that the idea of applying the method of paper [3] to the solution of spatial mixed problems for elastic layers was suggested by I.I.Vorovich.

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